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BY

MERRILL M. FLOOD

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SYSTEM
DEVELOPMENT
CORPORATION
2500 COLORADO AVE.
SANTA MONICA
CALIFORNIA
90406

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Abstract

John von Neumann (1953) has discussed a zero-sum two-person game and he has shown how the extreme optimal strategies for one of the players (the hider) can be calculated by solving a related assignment problem. We now offer an alternative treatment of the problem that is simpler and easily yields optimal strategies for both players.

THE HIDE AND SEEK GAME OF VON NEUMANN

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INTRODUCTION

John von Neumann (1953) has discussed a zero-sum two-person game and he has shown how the extreme optimal strategies for one of the players (the hider) can be calculated by solving a related assignment problem. We now offer an alternative treatment of the problem that is simpler and easily yields optimal strategies for both players.

THE GAME

Two players, a hider and a seeker, are given the n^2 values of a square array $||g_{ij}||$ of positive rational numbers. The hider chooses a cell (row and column indexes) and the seeker chooses a line (row or column index), each in ignorance of the choice made by the other player. If the seeker chooses a line that includes the cell chosen by the hider then the hider pays the seeker the amount for that cell, otherwise he pays 0. Thus, if the hider chooses cell (α,β) he pays $g_{\alpha\beta}$ to the seeker if and only if the seeker chooses row (α) or column (β) as his line. This completes one play of the game.

STRATEGIES AND VALUE

The hider has n^2 pure strategies corresponding to the n^2 cells. We let his mixed strategy be $h = (p_{ij})$ where $\Sigma_{ij}\, p_{ij} = 1$, and where p_{ij} denotes the probability that he hides in cell (i, j).

The seeker has 2n pure strategies corresponding to the 2n lines. We let his mixed strategy be $s = (p_i, q_i)$, where $\Sigma_i (p_i + q_i) = 1$, and where p_i denotes the probability that he seeks in row i and q_i in column i.

The expected value of the payoff from one play of the game, for the seeker, is $V(h, s) = \Sigma_{ij} p_{ij} g_{ij} (p_i + q_j)$.

We shall solve the game by exhibiting specific values p^*_{ij}, p^*_i, q^*_j, V^* that satisfy the relation:

1) $\underset{s}{\text{Max}}\ V(h^*, s) = \underset{h}{\text{Min}}\ V(h, s^*) = V^*$.

The value of the game is V^*, for the seeker, and $-V^*$ for the hider.

<u>ASSIGNMENT THEORY</u>

The assignment problem is to find a permutation of the columns of a square matrix, whose elements are rational numbers, that minimizes its trace†. Many solutions to this problem have been published. The Hungarian Method of H. W. Kuhn (1955) is the one we favor. The interested reader can find one version of this method in our earlier paper (Flood, 1961). We shall make use of some theoretical properties of this method of solution, and record them now for present purposes.

We let $I = (i_1, i_2, \ldots, i_n)$ and $J = (j_1, j_2, \ldots, j_n)$ represent column permutations of a square matrix of order n. Thus, I carries column r into column i_r and J carries column r into column j_r, where the n distinct elements of I, and J, are the first n positive integers. Therefore, I solves the assignment

† The trace of a square matrix is the sum of its main diagonal elements.

problem with matrix $||g_{ij}||$ if and only if, for every J, we have the relation

$$\Sigma_\alpha g_{\alpha i_\alpha} \equiv g_{1i_1} + g_{2i_2} + \dots + g_{ni_n} \le \Sigma_\alpha g_{\alpha j_\alpha} \quad .$$

The Hungarian Method yields a solution permutation I, and also yields values for 2n quantities u_i and v_i, that satisfy the following relations:

2) $g_{ij} + u_i - v_j \ge 0$, for $i, j = 1, 2, \dots, n$,

3) $g_{\alpha i_\alpha} + u_\alpha - v_{i_\alpha} = 0$, for $\alpha = 1, 2, \dots, n$.

HIDE AND SEEK GAME THEORY

We shall show how optimal strategies h^* and s^* can be written directly in terms of a solution to the assignment problem with matrix $||-1/g_{ij}||$.

We let J denote a solution of this assignment problem, and rewrite relations 2) and 3) for this case as follows:

4) $(-1/g_{ij}) + x_i - y_j \ge 0$,

5) $(-1/g_{\alpha j_\alpha}) + x_\alpha - y_{j_\alpha} = 0 \quad .$

We also define a quantity E by the following relations:

6) $(1/E) \equiv \Sigma_\alpha(1/g_{\alpha j_\alpha}) = \Sigma_\alpha(x_\alpha - y_\alpha) \quad .$

Finally, we define h^* and s^* by the following relations:

7) $p^*_{\alpha j_\alpha} \equiv E/g_{\alpha j_\alpha}$, all other $p^*_{ij} = 0$,

8) $p^*_\alpha \equiv E(x_\alpha - \min_i x_i)$, $q^*_\alpha \equiv E(\min_i x_i - y_\alpha) \quad .$

<u>Theorem.</u> The hide and seek game with matrix $||g_{ij}||$ has optimal strategies h^* and s^*, as defined in 4) - 8), and the value of the game to the hider is $V^* = E$.

<u>Proof.</u> We note that

$$\underset{s}{\text{Max}}\ V\ (h^*, s) = \underset{s}{\text{Max}}\ \Sigma_\alpha\ (E/g_{\alpha j_\alpha})\ g_{\alpha j_\alpha}\ (p_\alpha + q_{j_\alpha}) = E\ .$$

Also, using 4) and 5), that

$$\underset{h}{\text{Max}}\ V(h, s^*) = \underset{h}{\text{Max}}\ \Sigma_{ij}\ p_{ij}\ g_{ij}\ (x_i - y_j)\ E \geq \underset{h}{\text{Max}}\ \Sigma_{ij}\ p_{ij}\ E = E.$$

It remains to show that h^* and s^* are mixed strategies. Obviously, since $g_{ij} > 0$, the following quantities are non-negative: E, p^*_{ij}, p^*_α. Since, by (4), $x_i - y_j \geq (1/g_{ij}) > 0$ it follows immediately that $q^*_\alpha \geq 0$. Next, $\Sigma_{ij}\ p^*_{ij} = \Sigma_\alpha(E/g_{\alpha j_\alpha}) = 1$. Finally, $\Sigma_\alpha\ (p^*_\alpha + q^*_\alpha) = \Sigma_\alpha E(x_\alpha - y_\alpha) = \Sigma_\alpha E(x_\alpha - y_{j_\alpha}) = 1$. This completes our proof. We conclude with a simple illustrative example.

Numerical Example†. We apply these results to solve the following 2x2 hide and seek game:

$$\left|\left|g_{ij}\right|\right| \equiv \left|\left|\begin{matrix} 2 & 10 \\ 1 & 2 \end{matrix}\right|\right|, \text{ and } \left|\left|-1/g_{ij}\right|\right| = \left|\left|\begin{matrix} -1/2 & -1/10 \\ -1 & -1/2 \end{matrix}\right|\right| .$$

Obviously $J = (21)$, since $-11/10 < -1$, and then clearly

$x = (1/2, 1), y = (0, 2/5)$ satisfy 4) and 5).

Since $E = 10/11$, the non-zero values of p_{ij}^* are: $p_{12}^* = 1/11$ and $p_{21}^* = 10/11$.

Finally, $p_1^* = 0$, $p_2^* = 5/11$, $q_1^* = 5/11$, and $q_2^* = 1/11$. It is easily verified that $V(h^*, s) = 10/11$ and we find that $V(h, s^*) = (10/11) + (2/11)\, p_{22} \geq 10/11$.

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† Historical Note. These results were first obtained in late 1955, and presented in various lectures. We recently programmed the procedure (in JOVIAL) for the Q-32 time-shared computer at System Development Corporation, to provide a demonstration game on this system, because the hide and seek game is complex enough to be interesting to players but easily solved on the Q-32.

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